Postbuckling Analysis Using a General-Purpose Code

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A new capability for solving postbuckling problems in shell structures is described. The matrix theory to adapt Newton's method to nonlinear finite element shell analysis is outlined first. The matrix theory is directed at writing consistent linear algebraic equations for problems where the tangent stiffness matrix is singular or nearly singular. The matrix theory suggests a change of variables as part of the usual iterative procedure in Newton's method. The change of variables is shown to be feasible for introduction into the algorithm programmed in general-purpose codes for the finite element analysis of structures. Numerical results from a new option that has been programmed in an existing general-purpose code are presented. The analysis of shell structures for collapse and for branching at bifurcation loads is illustrated by the numerical examples.

Nomenclature

a	= length of rectangular plate
b	= width of rectangular plate
$D(\mu)$	= diagonal matrix of eigenvalues of matrix K
E	= residual error vector
$F(X,\lambda)$	= vector function of X and λ
$F_{i,\lambda}$	= partial derivative of the vector $F(X,\lambda)$ with respect to λ
h	= thickness of spherical cap
H	= rise of spherical cap
K	= tangent stiffness matrix
$K_{ij}, L_{ijk}, M_{ijkl}$	= coefficients in Taylor's series expansion of $F(X,\lambda)$
$K'_{rr}, K'_{rm}, K'_{mm}$	= submatrices of transformed stiffness matrix
N_x, N_{cr}	= in-plane compressive stress resultant in plate theory and classical buckling stress resultant, respectively
P	= elementary rectangular permutation matrix
q	= vector of modal amplitudes
$Q_{11}, Q_{12}, Q_{22}, Q_{66}$	= orthotropic plate material properties
R	= remainder term in Taylor's series expansion of $F(X,\lambda)$
u_i, U	= ith eigenvector of matrix K and matrix of eigenvectors, respectively
$\bar{u}, u_{cr}$	= average end shortening of compressed plate and end shortening at classical buckling load, respectively
$X, \Delta X$	= solution vector and correction of solution vector, respectively
Y	= transformed solution vector
z	= vector in partitioned form of vector Y
$\lambda, \Delta\lambda$	= load parameter and correction of load parameter, respectively
μ_i	= ith eigenvalue of matrix K
μ	= geometrical parameter in nondimensional shallow shell equations for a spherical cap
ν	= Poisson's ratio

Subscripts

i,j,k,l	= dummy subscripts
r	= rank of matrix K
m	= number of elements in vector q
n	= number of elements of vector X, also equal to the number of degrees of freedom in the finite element solution

Superscripts

T	= A^T denotes transpose of matrix A
$(\ )'$	= A' denotes the matrix resulting from some numerical transformation of the matrix A

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Introduction

GENERAL-PURPOSE finite element codes in use today for nonlinear structural analysis are slow to converge or fail to converge near the limit or bifurication points. This paper presents an adaptation of Newton's method that improves convergence near critical points where the linear form of Newton's method, the algorithm in most finite element codes, is ineffective. The adaptation is suitable for programming as an option in general-purpose codes for use in postbuckling analysis.

The adaptation of Newton's method retains certain higher-order terms that are dropped in the usual linearized iteration. The terminology, "adaptation of Newton's method," is used here for this variant of Newton's method rather than "modification of Newton's method" to prevent confusion with the form of Newton's method that does not update the linear operator at each iteration step. The computer logic in the adaptation extends theoretical and numerical results of Thurston[1-5] for systems of nonlinear differential equations to analogous results for systems of nonlinear algebraic equations generated by the finite element method.

For boundary value problems modeled by nonlinear differential equations, the adaptation of Newton's method seeks "compatible" solutions of a sequence of linear problems, where the homogeneous linear differential operators have nontrivial solutions satisfying the boundary conditions. The analogous adaptation of Newton's method for nonlinear finite element analysis is directed at solving "consistent" linear algebraic equations for cases where the linear operator, the tangent stiffness matrix, is singular or nearly singular. The singular linear operators, matrix or differential, point to the existence of multiple solutions and provide a method for computing multiple solutions. Therefore, the adaptation of Newton's method considered here is more general than methods that move along a single solution path.

For collapse analysis at isolated limit loads, the consistent equations solved by the adaptation of Newton's method are somewhat similar to the consistent incremental equations solved by Riks,[6] Crisfield,[7] Bergan,[8] and others where the linear algebraic equations are augmented by making the load parameter a dependent variable. For postbuckling solutions characterized in the analysis by bifurcation points, the adaptation represents a new approach for general-purpose codes that is sufficiently general so it can solve postbuckling problems exhibiting modal interaction. The theoretical basis for numerical analysis of bifurcation problems can be found in the book by Thompson and Hunt[9] and in the work of Koiter.[10] Reference 10 is especially relevent to the adaptation of Newton's method as used in this paper. The theory for different numerical solution methods for solving nonlinear problems has recently been reviewed by Riks.[11]

Koiter's theory is usually presented[12] in the context of perturbation theory rather than Newton's method. Since the perturbation method was developed at the turn-of-the century as an extension of linear vibration theory to the computation of periodic nonlinear solutions, the method does not lend itself directly to the matrix computations usually programmed for finite element analysis of shell structures. Haftka et al.[13] and Bresseling[14] have written computer codes based on perturbation theory for frame structures. Carnoy[15] has presented theory for the finite element analysis of shell structures based on Koiter's perturbation theory; however, the numerical analysis starts with the tangent stiffness matrix that is the discrete linear operator in Newton's method. Carnoy uses the eigenvectors of the tangent stiffness matrix corresponding to small eigenvalues near critical loads to generate a reduced nonlinear problem. The solution to the reduced problem is asymptotic to the solution of the nonlinear algebraic equations of the complete discrete model.§

The body of this paper is mainly concerned with presenting the adaptation of Newton's method as a practical option for general-purpose finite element codes. Computational procedures are emphasized instead of formal numerical analysis. The main feature of the adaptation, discussed in the next section, is a change of variables. The change of variables is used as a practical means of partitioning the problem to arrive at consistent linear algebraic equations at each iteration step. The linear form of Newton's method is reviewed first to put the change of variables into the context of the usual iteration sequence found in most solution algorithms. Next, the change of variables is defined. A subset of the new variables are modal variables. To achieve consistent equations, higher-order terms in a small number of modal variables are retained in the analysis, along with an unknown increment in the load parameter.

Following the section outlining the adaptation of Newton's method, the computer implementation of the change of variables is treated in more detail. The adaptation was implemented in the STACSC-1 general shell analysis code[16] and some of the changes in computer logic for the code are included. Finally, numerical examples applying the adaptation are presented.

§The authors are indebted to an anonymous reviewer for calling attention to Carnoy's paper, which has much in common with the present paper, both in the general approach to the problems of postbuckling analysis and in some specific aspects of the numerical analysis.

Adaptation of Newton's Method

This section describes the adaptation of Newton's method. The general approach is summarized. Details and background of the approach are given in the next section. Newton's method is reviewed to put the adaptation into the context of the normal iterative procedure. Finally, the changes in the algorithm required by the adaptation are examined in more detail, primarily changes produced by an equivalence transformation. The changes are shown to be suitable for programming into a general-purpose finite element code.

General Approach of the Adaptation

The finite element equations of the discretized structure can be cast in the form

$$F(X,\lambda)=0 \tag{1}$$

where X is an n-directional vector and λ is a load parameter. A typical iteration step in Newton's method solves

$$K\Delta X=-F \tag{2}$$

where ΔX is a correction vector to the current approximation for X. The residual error vector F and the tangent stiffness matrix K are evaluated at the current approximation for X. When K is singular or nearly singular, the correction ΔX from the solution of Eq. (2) will not be small, even though X is a good approximation and the residual error vector F is small. When K is nearly singular, the adaptation introduces a change of variables before solving Eq. (2),

$$\Delta X=TY \tag{3}$$

$$Y^T=[z_r{:}q_m]^T, \qquad r=n-m \tag{4}$$

After the change in variables and premultiplication by T^T, the equations that replace Eq. (2) in the iteration are

$$\begin{bmatrix} K'_{rr}K'_{rm} \\ (K'_{rm})^T K'_{mm} \end{bmatrix} \begin{bmatrix} z_r \\ \hline q_m \end{bmatrix} = -T^TF-T^TR \tag{5}$$

The general features of the adaptation as summarized in Eqs. (3-5) are the following:

1) The transformation matrix T is an $n\times n$ nonsingular matrix. Since T has an inverse, the transformation is called an equivalence transformation; the transformed problem has as many degrees of freedom as the original problem.

2) The matrix T is determined by the tangent stiffness matrix K. Several columns of T are eigenvectors of K corresponding to m small eigenvalues of K. The remaining submatrix of T is an elementary permutation matrix.

3) The form of the transformation matrix T partitions the unknown vector Y into a set of modal variables q and into variables z that are a permuted subset of the components of the original unknown vector ΔX. The $r\times r$ submatrix K'_{rr} of the transformed stiffness matrix is the matrix obtained by striking out m rows and the m corresponding columns of the original stiffness matrix K. The matrix K'_{rr} has an inverse; the submatrix K'_{mm} is singular or nearly singular.

4) The term R in Eq. (5) denotes higher-order terms in Y that are usually neglected at each iteration step of Newton's method and that appear as the residual error vector F in the next iteration step. Only higher-order terms in the modal variables q are retained in R; the variables z are then eliminated from Eqs. (5) to give a reduced nonlinear problem in the modal variables q.

5) The load parameter λ is also incremented in the residual error term F and allowed to vary in the search for real solutions of the reduced nonlinear problem, which are the only solutions of physical interest. The reduced nonlinear

problem is solved for either an exact or approximate solution for q. Back substitution completes the computation for ΔX. In the special case $m=1$, a single modal variable is prescribed and the reduced problem is solved for the correction to the load parameter.

6) The iteration step can be repeated as in the usual linear form of Newton's method. The stiffness matrix K, its submatrix K'_{rr}, and/or the transformation matrix T can be updated at each iteration step or stored as in the modified form of Newton's method. The overall solution strategy is to make Eqs. (5) "consistent" at each iteration step, in the same sense that consistent equations are defined in linear algebra.

Review of Newton's Method

The motivation for the change in variables defined by Eq. (3) is found by reviewing Newton's method. In reviewing the background theory, it is convenient to think of the tangent stiffness matrix K as a singular matrix with rank r equal to the rank of submatrix K'_{rr} in Eqs. (5). However, the practical application of the change of variables assumes only that r is not larger than the rank of K.

At the kth iteration step in Newton's method X_{k-1} is the current approximation for X, the exact solution that satisfies the nonlinear problem, and ΔX is the correction to be determined,

$$X=X_{k-1}+\Delta X \tag{6}$$

The nonlinear Eq. (1) can be rewritten by writing a Taylor series with remainder and by keeping the linear terms in components of ΔX on the left-hand side of the resulting equations,

$$K(X_{k-1},\lambda)\Delta X=-F(X_{k-1},\lambda)-R(X_{k-1},\Delta X,\lambda) \tag{7}$$

where K is the tangent stiffness matrix and R the remainder vector containing sums of quadratic and other higher-order terms in the components of ΔX.

Since Eqs. (7) are nonlinear in ΔX and cannot be solved directly, the linear form of Newton's method drops the remainder vector R and replaces the exact Eqs. (6) and (7) with the iterative sequence,

$$K(X_{k-1},\lambda)\Delta X_k=-F(X_{k-1},\lambda) \tag{8a}$$

$$X_k=X_{k-1}+\Delta X_k \tag{8b}$$

The residual error vector F, in each iteration step after the first, contains the remainder terms dropped during the previous iteration,

$$F(X_k,\lambda)=R(X_{k-1},\Delta X_k,\lambda) \tag{9}$$

Since the residual error is quadratic in the last correction, the linear form of Newton's method converges rapidly unless the tangent stiffness matrix K on the left-hand side of Eqs. (8a) is nearly singular, making the computed correction ΔX_k large even in cases where the residual error vector F on the right-hand side is small.

However, the exact correction is not necessarily large. The limiting case occurs when matrix K in Eqs. (8) is singular. The correction can be small if the equations are "consistent," as defined in linear algebra. A set of n simultaneous equations in n unknowns is consistent if the rank of the coefficient and the augmented matrices are equal. The latter matrix is obtained by augmenting the coefficient matrix by one column consisting of the vector of the right-hand side. The residual error vector F, in general, will not produce a consistent set of equations and additional analysis is required.

The adaptation of Newton's method seeks a consistent set of equations by retaining selected terms of the remainder $R(X_{k-1},\Delta X_k,\lambda)$ that are usually dropped in going from the exact Eq. (7) to the iterative sequence [Eqs. (8)]. Retaining certain higher-order terms and dropping others was first suggested for postbuckling analysis by Koiter[10] for continuum solutions along with theoretical justification for expecting convergence. No attempt is made here to prove convergence of the analogous discrete theory, since the concern here is the application of the theory to a general finite element solution.

The first step in making the equations consistent in the limiting case is to determine the rank of the singular stiffness matrix. The theoretical definition of rank involves minor determinants of the matrix. A practical measure of rank is obtained from the number of zero eigenvalues. The connection between eigenvalue analysis and the analysis of rank provides the motivation for the form of the equivalence transformation [Eq. (3)] used in the adaptation of Newton's method. In the practical application, the equivalence transformation is made when the eigenvalues are small rather than only for the limiting case defined by zero eigenvalues. The form of the equivalence transformation suggested by the eigenvalue analysis is considered next.

Equivalence Transformation

The change of variables at each iteration step in the adaptation of Newton's method is

$$\Delta X_k=T_k Y_k \tag{10}$$

The transformation matrix T used here is derived from eigenvectors corresponding to small eigenvalues of the tangent stiffness matrix K, which in this paper is assumed to be symmetric. The transformation matrix is partitioned

$$T=[P_{nr}:U_{nm}] \tag{11}$$

The m columns of submatrix U are m eigenvectors of K,

$$KU=UD(\mu) \tag{12}$$

The set of eigenvalues μ are the m smallest eigenvalues in absolute value. The exact number m is arbitrary, except m is assumed to be greater than the nullity of K when K is singular.

The submatrix P is chosen to be an elementary permutation matrix with zero elements except for one nonzero element equal to unity in each column. Since P is rectangular with n rows and $r=(n-m)$ columns, there are m row vectors of P that are zero. The row numbers are selected to correspond with row numbers of maximal elements of U, thereby preventing the occurrence of zero row vectors for T. A zero row vector would make T singular. The matrix T must be square and nonsingular for Eq. (10) to be an equivalence transformation, so that any transformed problem written in terms of Y is equivalent to the original problem in X.

The unknown vector Y introduced in Eq. (10) is partitioned to be conformable with the submatrices of T,

$$Y^T=[z_r,q_m]^T \tag{13}$$

The form of submatrix P identifies the vector z as a permuted subset of the components of ΔX and the definition of U identifies the components of q as modal variables. This identification helps in the physical interpretation of the numerical analysis of the transformed physical problem.

Transformed Nonlinear Problem

Introduction of the change in variables in Eq. (10) into the nonlinear problem obtains a transformed nonlinear problem. The nonlinear problem is partitioned into two coupled problems. By temporarily dropping nonlinear terms in the z variables, a nonlinear problem is derived in the modal variables q. After solving the reduced nonlinear problem for the q variables, the z variable can be determined. Because of dropping higher-order terms in z, the solution for Y is approximate and the procedure must be put in an iteration loop similar to the usual iteration sequence in Newton's method.

The exact form of the nonlinear transformed problem is obtained by substituting the change in variable [Eq. (3)] into Eqs. (6) and (7) to give

$$X = X_{k-1} + TY \tag{14}$$

$$K(X_{k-1},\lambda)TY = -F(X_{k-1},\lambda) - R(X_{k-1},Y,\lambda) = -E \tag{15}$$

Premultiplication of Eqs. (15) by the transpose of T makes the transformed stiffness matrix K' symmetric

$$K'Y = -E' \tag{16}$$

where

$$K' = T^TKT \tag{17a}$$

$$E' = T^TF(X_{k-1},\lambda) + T^TR(X_{k-1},Y,\lambda) \tag{17b}$$

The adaptation of Newton's method is to drop all terms in the z components of Y in the nonlinear remainder term R in Eq. (17b), while retaining modal variables q. The adaptation replaces the exact E' in Eq. (16) with the approximation

$$E' = E'(X_{k-1},q,\lambda) = T^TF(X_{k-1},\lambda) + T^TR(X_{k-1},q,\lambda) \tag{18}$$

Because of the approximation for E' [Eq. (18)], the solution of Eq. (16) for Y is not exact. An iterative sequence is required with $Y = Y_k$, the approximate solution of Eq. (16) and

$$X_k = X_{k-1} + T_kY_k \tag{19}$$

replacing Eq. (14). As in the linear form of Newton's method, X_k replaces X_{k-1} at the next iteration step.

The reason for expecting rapid convergence of the iterative sequence, including cases where the linear form of Newton's method diverges, is found by examining the partitioned transformed stiffness matrix

$$K' = T^TKT = \begin{bmatrix} K'_{rr} & K'_{rm} \\ (K'_{rm})^T & K'_{mm} \end{bmatrix} \tag{20}$$

The partitioning is conformable with the partitioning of Y and of T in Eq. (11) and

$$K'_{rr} = P^TKP \tag{21a}$$

$$K'_{rm} = P^TKU \tag{21b}$$

$$K'_{mm} = U^TKU \tag{21c}$$

The properties of the elementary submatrix P produce the submatrix K'_{rr} by deleting m rows and the m corresponding columns of K. Since the submatrix U contains eigenvectors of K [Eq. (12)], K'_{mm} and K'_{rm} are given by

$$K'_{mm} = D(\mu) \tag{22a}$$

$$K'_{rm} = P^TUD(\mu) \tag{22b}$$

The postmultiplication by the diagonal matrix $D(\mu)$ in Eqs. (22) makes an eigenvalue μ appear as a factor in each of m columns (rows) of the transformed stiffness matrix K'. For the limiting case of m eigenvalues equal to zero, the m columns (rows) are zero and K' is singular with nullity m.

However, even for the limiting case of singular K', the submatrix K'_{rr} has an inverse. The inverse is used to solve the partitioned form of Eq. (16) where the right-hand side of the equation is written by partitioning the approximation for E' [Eq. (18)] to be conformable with K',

$$K'_{rr}z_r + K'_{rm}q_m = -E'_r \tag{23a}$$

$$(K'_{rm})^Tz_r + K'_{mm}q_m = -E'_m \tag{23b}$$

Eliminating z from Eqs. (23) leads to a nonlinear problem in q with m unknowns,

$$z_r = -(K'_{rr})^{-1}E'_r - (K'_{rr})^{-1}K'_{rm}q_m \tag{24a}$$

$$K''_{mm}q_m = -E''_m \tag{24b}$$

where

$$K''_{mm} = K'_{mm} - (K'_{rm})^T(K'_{rr})^{-1}K'_{rm} \tag{25a}$$

$$E''_m = E''_m(X,q) = E'_m - (K'_{rm})^T(K'_{rr})^{-1}E'_r \tag{25b}$$

Once the modal nonlinear equations [Eq. (24b)] are solved for q, the iteration step is completed by the explicit computation of z in Eq. (24a) and of X_k in Eq. (19).

However, the nonlinear modal equations for determining q must be written explicitly before they can be solved. Up to this point, the remainder terms $R(X,q,\lambda)$ have been written symbolically. The next section describes the actual computer operations required to arrive at an explicit form of Eqs. (24).

Computer Implementation

The adaptation of Newton's method presented here requires some numerical computations in addition to those in the linear form of Newton's method. This section discusses the additional operations that must be programmed to make the adaptation an option in an existing general-purpose finite element computer code. Some specific information is also included on a new option in the STAGS C-1 code,[16] called the transformation processor (TP) option. Most of the comments on computer implementation are general. Comments that apply only to the new TP option in the STAGS C-1 code will be so identified.

The operations to write the transformed stiffness matrix are relatively simple to program, simpler than the formal matrix theory indicates. The higher-order remainder terms are not usually calculated by finite element codes. The terms are computed indirectly in the TP option.

Transformed Stiffness Matrix

The change of variables is a linear equivalence transformation. The matrix operations for computing the transformed stiffness matrix [Eqs. (20)] are standard matrix multiplications. The main consideration in programming the transformed stiffness matrix is to take advantage of the sparseness of the submatrix P in the transformation matrix T [Eq. (11)]. To form T, the m eigenvectors in the submatrix U must be computed. Inverse subspace iteration or other standard methods for computing eigenvectors corresponding to small eigenvalues are suitable. The eigenvalues and eigenvectors need not be computed to high precision; however, use of the equivalence transformation implies that the eigenvectors are linearly independent.

The submatrix U is stored for later use. The submatrix P is not stored, only the row numbers of the nonzero element in each column are stored for use in programming the permutations indicated in the theory by multiplication by matrix P.

Because of the form of P, the elements of the submatrix K'_{rr} of the transformed stiffness matrix are permuted elements of the original stiffness matrix K. Since computing, storing, and factoring the sparse stiffness matrix K is a major part of the computer logic for a general-purpose code, using the elements of K to factor K'_{rr} in Eqs. (24) is a key part of making it feasible to add the adaptation of Newton's method to an existing general-purpose code.

The remaining submatrices in the tranformed stiffness matrices must be computed and stored. The computations may use Eqs. (21) or the simpler set [Eqs. (22)]. The latter set applies only when each column of matrix U is an exact orthonormal solution of Eqs. (12). When the submatrix U is not updated at every iteration step, the matrix multiplications in Eqs. (21) must be used.

In the TP option, an existing branch of the STAGS C-1 code is utilized to generate the eigenvectors in U. The branch computes eigenvalues that are multipliers of "the nonlinear prestress."[16] The branch uses an inverse subspace iteration subroutine to compute eigenvalues and eigenvectors. The eigenvectors are normalized to a maximum element of unity. Since the eigenvectors are for a generalized eigenvalue problem and the eigenvectors are not orthonormal, the TP processor computes using Eqs. (21).

The submatrix K'_{rr} is embedded in the stored $n \times n$ assembled stiffness matrix in the TP option. During the formation of the matrix from the element stiffness matrices, the bandwidths of m equations corresponding to m nodal variables, identified in the analysis by matrix P, are set equal to one and a zero is stored on the main diagonal. The original numbering system for nodal variables is retained. The logic in the subroutine for Cholesky factorization of matrices is unchanged. In the factorization of K'_{rr}, when an equation is reached with a bandwidth of one and a zero on the main diagonal, a zero is stored in the solution vector for the corresponding dummy variable.

While writing and factoring the transformed stiffness matrix is straightforward, writing the right-hand side of Eqs. (24) for a numerical solution is more complicated. The remainder term $R(X,q,\lambda)$ must be written in vector notation.

Nonlinear Remainder Terms

In the formal matrix analysis, the remainder term R has been written as a mean value of a second-order differential form.[17] For actual computations, the remainder term can be approximated by extending the Taylor series expansion of Eq. (7) through cubic terms and dropping the remainder term for the higher-order expansion. The formal expansion is in powers of the components of the new variables Y [Eq. (10)]. However, only powers of the nodal variables q are retained in the quadratic and cubic terms.

The expansion through cubic terms is written in indicial notation

$$F(X,Y,\lambda) = F_i(X,\lambda) + K_{ij}Y_j + L_{ijk}Y_j y_k + M_{ijkl}Y_jY_kY_l \quad (26)$$

The elements K_{ij} are elements of the matrix product KT, which can be obtained using the chain rule of differentiation and the linear change of variables [Eq. (10)]. With the K_{ij} coefficients known as part of the operation of writing the transformed stiffness matrix, the approximate values of coefficients L_{ijk} and M_{ijkl} can be computed numerically. Coefficients of a specific modal variable q_p are computed by multiplying the eigenvector u_p by a small value of q and computing the residual vectors $F(X+qu_p,\lambda)$ and $F(X-qu_p,\lambda)$ to obtain

$$F_i(X+qu_p,\lambda) = F_i(X,\lambda) + qK_{ip} + q^2L_{ipp} + q^3M_{ippp} \quad (27a)$$

$$F_i(X-qu_p,\lambda) = F_i(X,\lambda) - qK_{ip} + q^2L_{kpp} - q^3M_{ippp} \quad (27b)$$

The coefficients L_{ipp} and M_{ippp} are stored as vectors with subscript i,

$$L_{ipp} = [F_i(X+qu_p,\lambda) - 2F_i(X,\lambda) + F_i(X-qu_p,\lambda)]/(2q^2) \quad (27c)$$

$$M_{ippp} = [F_i(X+qu_p,) - 2qK_{ip} - F_i(X-qu_p,\lambda)]/(2q^3) \quad (27d)$$

and appear in the numerical analysis multiplying quadratic and cubic powers of the modal variable q_p as part of R, starting with Eqs. (18). The coupling of modal variables can be computed in a similar fashion; L_{ips} and M_{ipss} can be computed from computing $F(X \pm q_p u_p \pm q_s u_s,\lambda)$ plus Eqs. (27).

Correction in Load Parameter

The number of higher-order terms required in the numerical analysis is problem dependent. Only enough terms are required to make Eqs. (24) consistent for the limiting case where K is singular. In the practical application of the adaptation, it is necessary only to continue the iterative procedure by finding small real solutions of the nonlinear modal problem [Eq. (24b)]. The solutions of the modal equations in the early steps of the iteration can be approximate, but to be of physical interest they must be real.

In order to obtain only real solutions, the load parameter λ can be varied in the Taylor series expansion for the remainder term R. Only linear terms are retained in the expansion,

$$F_i(X,\lambda+\lambda\Delta) = F_i(X,\lambda) + F_{i,\lambda}\Delta\lambda \quad (28a)$$

The vector $F_{i,\lambda}$ is computed by using a forward difference operator

$$F_{i,\lambda} = [F(X,\lambda+\Delta\lambda) - F(X,\lambda)]/\Delta\lambda \quad (28b)$$

and retained in the numerical analysis multiplying the unknown correction in the load parameter as another term in the remainder R.

The derivative of the residual with respect to the load parameter $F_{i,\lambda}$ has practical significance in the computer implementation of the solution of the transformed problem [Eqs. (24)]. In the case where the nullity of the singular tangent stiffness matrix is one, the derivative is the only higher-order term needed in the computations. The numerical analysis for this special case is outlined in the following paragraphs. The numerical examples in the next section illustrate problems where the special case of nullity equal to one has physical significance.

Case of Nullity Equal to One

For many postbuckling problems, the nullity of the singular tangent stiffness matrix is one at critical points. The critical points can be either isolated limit points or isolated bifurcation points. In the numerical analysis, the solution of the transformed problem [Eqs. (24)] requires only one stored scalar element of the residual error vector to be zero after the forward sweep in the factorization of the algebraic equations, which is numerically equivalent to an orthogonality relation in the modal analysis. The residual error vector in the analysis is stored as two vectors in the computer code. The remainder term R in Eq. (18) is computed explicitly by specifying the modal amplitude q_1 as input. Then, rather than starting the iteration at a known solution X of the nonlinear problem at a given load, the solution in the new option in STAGS C-1 is started with the zeroth approximation

$$X_0 = X + q_1 u_1 \quad (29)$$

The residual error vector for the zeroth approximation is computed and stored. The vector, viewed as the sum of Taylor's series, contains linear and higher-order terms in q_1, but the residual vector is small because $F(X,\lambda)$ is zero and the linear term in q_1 vanishes at the critical point. The reduce the small residual vector $F(X_0,\lambda)$ further, it is augmented by a differential vector containing a correction to the load parameter to be determined. Therefore, Eq. (18) contains two vectors in the numerical solution

$$E' = T^T F(X_0,\lambda) + T^T F_{i,\lambda}(X_0,\lambda)\Delta\lambda \qquad (30)$$

where the partial derivative with respect to the load parameter is computed from Eqs. (28) with X replaced by X_0.

After the factorization of K'_{rr} indicated by Eq. (24a), the single scalar Eq. (24b) has the form

$$K''_{mm} q_1' = F'(X_0,\lambda) + F'_{i,\lambda}(X_0,\lambda)\Delta\lambda \qquad (31)$$

The variable q_1' in Eq. (31) is a variable in $X_1 = T_1 Y_1$ and is the correction on the modal amplitude q_1 that appears in the zeroth approximation X_0 in Eq. (29). However, K''_{mm}, the coefficient of q_1', is known to be small, vanishing in the limiting case when $K(X,\lambda)$ is singular. Therefore, the modal amplitude q_1 is fixed and the correction q_1' is set equal to zero in Eq. (31). The resulting equation is satisfied by solving for $\Delta\lambda$.

Once $\Delta\lambda$ and q_1' are known, the two vectors in E_r' on the right-hand side of Eq. (24a) are superposed. The iteration step is completed by the explicit solution of Eq. (24a) for z_r. After completing the iteration step, the current approximation for X and λ can be improved by repeating the iteration while holding the modal amplitude q_1 fixed.

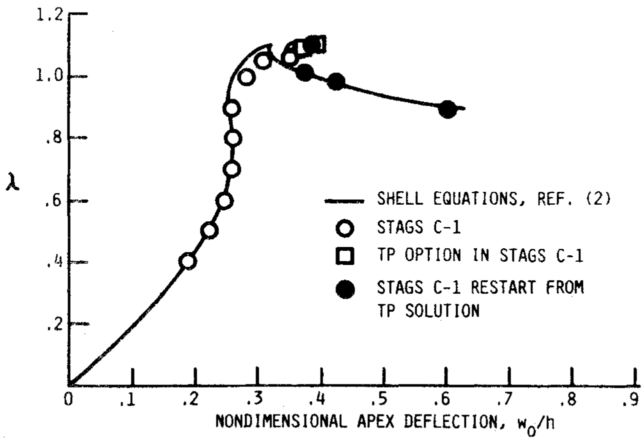

Fig. 1 Nondimensional pressure deflection curve for shallow spherical cap.

In the formal numerical analysis, Eq. (31) is part of a consistent set of linear algebraic equations when K''_{mm} and the right-hand side both vanish. In the practical application of the analysis, division of a small residual error by a small value of K''_{mm} is avoided by always making the right-hand side of Eq. (31) zero by correcting the load parameter.

The numerical results presented in the next section illustrate the use of Eq. (31) in the TP option. The results are for a physical problem exhibiting collapse and another with bifurcation points.

Numerical Examples

Two typical numerical examples are presented here. Each example is representative of a broad class of problems. The first is simple collapse analysis, where a generalized load deflection curve exhibits a local maximum load or limit point. The second is the analysis near bifurcation points where a primary load deflection curve is intersected by a postbuckling load deflection curve. The specific examples presented here are the collapse analysis for clamped spherical caps under hydrostatic pressure and the postbuckling analysis of flat plates in compression.

Limit Points

Since the linear form of Newton's method will not converge at a limit point, the adaptation of Newton's method is applied in the neighborhood of a limit point. The slow convergence of the linear form of Newton's method is a signal to switch to the adaptation of Newton's method.

The example of a limit load analysis presented here is the snap-through of a shallow spherical cap under hydrostatic pressure. The numerical results, shown in Fig. 1, are for a cap with a geometric parameter $\mu = [2H/h]^{1/2} \div [12(1-\nu^2)]^{1/4} = 8.1$. The load parameter λ for this case is the ratio of the applied hydrostatic pressure to the classical buckling pressure for a complete spherical shell. The cap is clamped at the outer edge. The solid curve in Fig. 1 is taken from Ref. 2 and is a plot of the load parameter λ as a function of the nondimensional deflection W_0/h, at the apex of the cap. The results in Ref. 2 are from the solutions of Reissner's equations for axisymmetric finite deflections of shells of revolution, specialized to the shallow shell problem. The limit load from the shallow shell equations is $\lambda = 1.13$.

The points plotted as circles in Fig. 1 are results using the linear form of Newton's method in the STAGS C-1 code. The linear algorithm converges rapidly in varying λ in increments of $\lambda = 0.4$-1.05 and converges slowly at $\lambda = 1.10$. The TP option computed the equivalence transformation based on the solution for X at $\lambda = 1.10$ and generated the results shown by the squares. The circles past the limit point are results from switching back to the linear algorithm in the STAGS C-1 code with the zeroth approximation for the first solution coming from the results from the TP option.

The finite element results give a good approximation to the results from shell theory considering that the apex of the

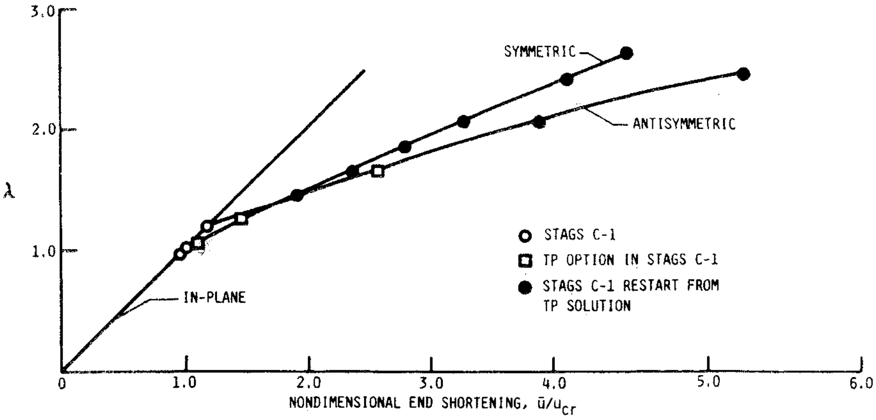

Fig. 2 Example of problem with discrete bifurcation points: inplane, symmetric, and antisymmetric solutions for a simply supported plate in compression.

cap is a singular point in spherical coordinates and that the maximum deflection for pressures near the collapse pressure is not at the apex and is approximately four times the apex deflection. Of more importance than discretization error in the context of this paper is the agreement between numerical results from the TP option and from the STAGS C-1 code. In theory, the equivalence transformation introduced by the adaptation of Newton's method does not change the discrete model. The numerical results for the cap problem verify this theoretical result.

The results from the TP option also agree with results, not shown in Fig. 1, from the option in STAGS C-1 for a "path length" analysis. This latter option is based on the continuation procedure of Riks.[6] The numerical analysis for the equivalence transformation and the "path length" computation are somewhat similar for the limit point analysis. The load parameter is treated as dependent variable in both methods, with one method using a modal variable as an independent variable and the other method using a path length.

The postbuckling analysis near isolated bifurcation points, the concept of following a unique solution path breaks down. However, the modal analysis provides a method for moving from one solution branch to another, as shown in the next example.

Bifurcation Points

For limit points, the current logic in the TP option for computing the load increment $\Delta\lambda$ for fixed modal amplitude has some similarities with the logic in other finite element codes for continuing solutions through limit points (e.g, Refs. 6-8). This logic fails for bifurcation points.

In the notation of this paper, the failure in continuation methods occurs in Eq. (31) because the divisor multiplying the correction to the load parameter is equal to zero. The coefficient vanishes because the partial derivative of the residual error is taken about a solution on a primary load deflection curve. The derivative on that branch is orthogonal to the eigenvector of the tangent stiffness matrix and corresponds to the first eigenvalue, also zero at the bifurcation point.

Results for a specific example of computation near bifurcation points are summarized in Fig. 2. The nonlinear problem is a simply supported flat plate of orthotropic material under an in-plane compressive load. The nondimensional material properties are $Q_{12}/Q_{11}=0.3$, $Q_{22}/Q_{11}=1.05$, and $Q_{66}/Q_{11}=0.45$ and the aspect ratio is $a/b=1.2$. Opposite edges of the plate remain straight and parallel at all loadings. The nondimensional end shortening is u/u_{cr}. The load parameter for this problem is $\lambda=N_x/N_{cr}$, where N_{cr} is the first buckling stress resultant, which corresponds to a buckling mode with transverse deflections that are symmetric about the centerline of the plate perpendicular to the load direction. The second buckling mode at $\lambda=1.187$ is antisymmetric about the same centerline.

After computing a solution on the primary load path at $\lambda=0.95$ using STAGS C-1, the TP processor computed the solutions on the postbuckling curves marked by squares in Fig. 2. The TP processor converged to symmetric solutions at $\lambda=1.056$ and 1.25 with q_1 equal to 0.5 times the plate thickness and q_1 equal to 1.0 times the thickness, respectively. For each solution, q_1 multiplied the first symmetric eigenvector to generate the zeroth approximation. The points marked by circles on the symmetric postbuckled curve are for a computer solution using the restart capability in STAGS C-1 with the solutions from the TP processor as zeroth approximations for the linear form of Newton's method.

The postbuckled solutions that are antisymmetric about the plate centerline are computed in the same manner as the symmetric solutions. With q_2 equal to the plate thickness and multiplying the second eigenvector (the first antisymmetric mode), the TP processor converged to the antisymmetric postbuckling solution at $\lambda=1.65$.

The postbuckling load deflection curves cross, suggesting the possibility that the symmetric solution may become unstable at some load higher than the antisymmetic load corresponding to the same end-shortening value. However, the STAGS C-1 solutions do not indicate any loss of stability by secondary buckling for this particular problem. The TP processor option remains to be tested for problems that exhibit modal interaction.

The equivalence transformation requires an eigenvalue analysis. The current version of the code uses the method of subspace iteration. An eigenvalue analysis requires more computations than a matrix factorization. Preliminary experience with the TP option indicates that fewer load steps are required to get into the postbuckling range than other methods of collapse analysis. The cost of more computations at one load to arrive at the equivalence transformation is balanced by the savings from less load steps. (The numerical examples were computed on a minicomputer. The TP option is now available in codes for a main-frame computer and a supercomputer.)

The two examples presented here indicate that the logic in the TP processor prescribing one modal variable and solving for the load increment, which makes the augmented linear equations [Eqs. (24)] consistent, is sufficiently general to be applicable to a large number of practical nonlinear problems.

Conclusions

An adaptation of Newton's method applicable to nonlinear structural analysis near collapse and bifurcation loads has been outlined. The adaptation introduces a change of variables, or equivalence transformation, into the iterative procedure used in general-purpose codes. The equivalence transformation is used to make the linear algebraic equations defined by the tangent stiffness matrix consistent for cases where the matrix is singular or nearly singular.

The equivalence transformation is nonsingular and models the original finite element problem. The transformation provides the theoretical insight of modal analysis and, at the same tiem, is practical to program for an existing general-purpose code. The transformation has been successfully introduced as an option in an existing general-purpose finite element code for nonlinear analysis of shell structures, STAGS C-1.

Numerical results are presented for collapse and postbuckling analyses near bifurcation points. The collapse analysis parallels the continuation methods that have been applied in other finite element codes. The capability for analysis of near bifurcation points as described here has not been previously available in a general-purpose code.

The equivalence transformation shows great potential for solving modal interaction problems. Interaction problems become important when optimization codes that satisfy buckling constraints provide a structural design with coincident buckling loads corresponding to different mode shapes. The nonlinear finite element analysis for predicting failure in these structures must consider multiple solutions of the finite element equations, which is now feasible in a general-purpose code by including the equivalence transformation described in this paper.

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